Abstract

A relatively new way of combining modal logics is to consider their products. The main application of these product logics lies in the description of parallel computing processes. Axiomatics and decidability of the validity problem have been rather extensively investigated and many logics behave well in these respects. In this paper we look at the product construction from a computational complexity point of view. We show that in many cases there is a drastic increase in complexity, e.g., all products containing the finite $S_5 \times S_5$ products as models have a $\text{nexptime}$-hard satisfaction problem. Products with a functional modality however do not lead to an increase in complexity. For the products $K \times S_5$ and $S_5 \times S_5$, we provide a matching upper bound.

Combining (modal) logics is a very active area, witness e.g., [4] and the book [1]. A rather special way of combining two modal logics is to consider their products. This approach started with [20], and has recently been developed in great detail in [5]. In temporal logic, products of two logics have been used to describe the temporal logic of intervals (cf. the “product treatment” of the system HS from [8] in [15]: Chapter 4 and the references therein). Almost all products of temporal logics are undecidable, sometimes the validities are not even recursively enumerable [8, 22]. Products of modal (and modal and temporal) logics have applications in the theory of parallel computing [18].

Here we are concerned with the general mathematical theory of products of modal logics, in particular the complexity of several natural decision problems, like the validity problem.

With respect to (Hilbert style) axiomatizability a lot of general results are obtained in [5], cf e.g., Theorem 5.7. That paper also contains decidability results for a large number of cases. The general trend for these results is that they are rather hard to prove, but become a lot easier if one of the logics is $S_5$, though even then the filtration arguments are rather involved, and lead to models whose size is in general double exponential in the length of the formula which is to be satisfied. The upper bounds we obtain from these proofs are very bad, in the general case (when none of the logics is $S_5$), the decision-algorithm is non-elementary, and when one of the logics is $S_5$ we only obtain a non-deterministic double exponential time upper bound for the satisfaction problem.

Questions concerning computational complexity, have hitherto not been addressed, and we will make a start here. The overall trend is that these logics have a very bad complexity for the satisfaction problem: in many simple cases it is $\text{nexptime}$-hard. Also, even if the satisfaction problem is decidable, the problem whether for a formula $\varphi$ there exists a model

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in which \( \varphi \) is true in every state might be undecidable (for instance in the product of \( K \) and \( K \)). A positive point is that products with a functional modality do not lead to an increase in complexity.

This rather short note by no means provides a thorough study of the whole field of product logics. Our aim is to put the (in our opinion, exciting question of) complexity of product logics onto the research agenda, and show in a sense the first steps on the complexity ladder (unfortunately they are put already pretty high). For this reason we will provide a list of the most challenging open problems in this area.

The paper is organized as follows. We start with the definition of product logics and recall some basic facts about them. In section 2, we prove lower complexity bounds using easily applicable tiling problems. In the next section, we find upper bounds for products where one of the components is rather simple, i.e., either \( S5 \) or a functional modality. Finally we recall some related work and results, and finish with a list of open problems.

1 Preliminaries

We start with the necessary preliminary definitions. We assume familiarity with standard modal logical notions and with the most well known modal logics like \( K, S4, S5 \), etc. For convenience we list the frame-properties for which the logics we will use are complete.

\[
\begin{align*}
(1) \quad & \forall xRxx & \text{reflexivity} \\
(2) \quad & \forall xyz((Rxy \land Ryz) \rightarrow Rxz) & \text{transitivity} \\
(3) \quad & \forall xyz((Rxy \land Rxz) \rightarrow (y = z \lor Ryz \lor Rzy)) & \text{weak linearity} \\
(4) \quad & \forall xy(Rxy \rightarrow Ryx) & \text{symmetry} \\
(5) \quad & \forall xyz((Rxy \land Rxz) \rightarrow y = z) & \text{partial functionality} \\
(6) \quad & \forall x \exists yRxy & \text{seriality}.
\end{align*}
\]

Table 1 characterises our logics.

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Table 1: Characterization of modal logics.

**Product logics.** Products of modal logics are defined in [5] as follows. The *product* \( \mathfrak{F} \times \mathfrak{G} \) of two standard modal frames \( \mathfrak{F} = (W, R) \) and \( \mathfrak{G} = (W', S) \) is the modal frame \( (W \times W', H, V) \), in which the relations are defined as follows

\[
(x, y)H(x', y') \iff xRx' \text{ and } y = y'.
\]
\[(x, y)V(x', y') \iff ySy' \text{ and } x = x'. \]

A product of two uni-modal frames leads to a bi-modal frame. We will use \(\Phi\) for the modality which uses the \(V\)- (for vertical), and \(\Theta\) for the modality which uses the \(H\)- (for horizontal) relation. Their meaning is defined in the standard way, i.e., \(\mathfrak{M}, w \models \Phi \varphi\) iff there exists a \(w'\) such that \(wVw'\) and \(\mathfrak{M}, w' \models \varphi\).

For \(K\) and \(K'\) both classes of modal frames, their product \(K \times K'\) is the class of frames
\[\{\mathfrak{F} \times \mathfrak{G} \mid \mathfrak{F} \in K \text{ and } \mathfrak{G} \in K'\}\].

If \(K = K'\) we also use the notation \(K^2\) to denote \(K \times K\).

For the well-known modal logics, like \(K, S4, S5\), etc, we use the notation \(K \times K\) and so on, to denote the product of the largest classes of frames for which these logics are complete.

Instead of binary products, one can of course take products of any dimension. As in [5], we will mostly concentrate on the binary case.

Products of modal logics are more related to first order logic than to standard modal logic. This shows already in the obvious standard translation for \(K \times K\) products, where we see that propositions denote binary relations instead of unary ones. Let \(\langle \cdot \rangle^{st(x,y)}\) be the following function from the bimodal language of \(\Phi\) and \(\Theta\) to first order logic
\[p^{st(x,y)} = P(x, y)\]
\[\langle \cdot \rangle^{st(x,y)}\] is a homomorphism for the boolean
\[(\Phi \varphi)^{st(x,y)} = \exists z(yVz \land \varphi^{st(x,z)})\]
\[(\Theta \varphi)^{st(x,y)} = \exists z(xHz \land \varphi^{st(z,y)})\].

Note that the range of the translation is first order logic with \(three\) variables, unlike the standard one-dimensional case where two variables suffice. It is an easy exercise to show that the translation is truth-preserving.

In fact the predicate calculus can be defined as a product of modal (S5) logics, and this is precisely what is done in cylindric algebra theory. Let \((D, I)\) be a first order model for a language containing only \(n\)-ary relation symbols (no equality, constants or functions). Let \(D^n\) denote the \(n\)-dimensional product model \((D^n, \equiv_i, I)_{i<n}\) where the \(\equiv_i\) relations are defined as
\[s \equiv_i t \text{ iff for all } j \neq i, s(j) = t(j).\]

Clearly the frame \((D^n, \equiv_i)_{i<n}\) is an \(n\)-dimensional power of the frame \((D, D \times D)\), and we can use it to interpret the modal language with \(n\) diamonds \(\Diamond_i\). As is easy to see, this modal language is just a syntactic variant of a restricted version of first order logic with \(n\) variables \(\{v_0, \ldots, v_{n-1}\}\). The restriction is that there are only \(n\)-ary relation symbols, and the only atomic formulas are of the form \(R(v_0, \ldots, v_{n-1})\). Now the bijection \(\langle \cdot \rangle^*\) can be given as
\[p^* = P(v_0, \ldots, v_{n-1})\]
\[\langle \cdot \rangle^*\] is a homomorphism for the boolean
\[(\Diamond_i \varphi)^* = \forall v_i \varphi^*\].

Now for all formulas \(\varphi\), for all models \(\mathfrak{M} = (D^n, \equiv_i, I)_{i<n}\), for all \(s \in D^n\),
\[\mathfrak{M}, s \models \varphi \iff (D, I) \models \varphi[s].\]

In particular, since the modal theories of S5\(^2 = \{\mathfrak{F}_0 \times \mathfrak{F}_1 \mid \mathfrak{F}_i = (W_i, R_i)\} with \(R_i\) an equivalence relation\) and of the class \(\{\mathfrak{F} \times \mathfrak{F} \mid \mathfrak{F} = (W, W \times W)\} \) for some set \(W\) are the same [5], we see that the
product $S_5^2$ is this restricted 2-variable fragment of the predicate calculus. For more on this modal treatment of first order logic, cf., [15].

If we consider products axiomatically we also meet this first order flavour in the axioms which are valid in every product frame

\[
\Phi \boxdot p \leftrightarrow \boxdot \Phi p \quad \text{commutativity}
\]
\[
\boxdot p \leftrightarrow \boxdot \Phi p \quad \text{confluence},
\]

which are well-known as commutativity of the quantifiers and quantifier exchange. On modal frames $(W, H, V)$ these Sahlqvist axioms correspond to the following first order conditions

\[
\forall xy (\exists z (xVz \land zHy)) \leftrightarrow \exists z (xHz \land zVw),
\]

(1)
\[
\forall xyz ((xHy \land xVz) \rightarrow \exists w (yVw \land zHw)),
\]

(2)

For $L_1$ and $L_2$, two axiomatically defined modal logics, let $[L_1, L_2]$ denote the logic obtained by adding the commutativity and the confluence axioms (and obviously extending the substitution rule to the whole new language) [5]:Definition 3.10. One of the main results of [5] (Theorem 7.12) is a sufficient condition on $L_1$ and $L_2$ ensuring that $[L_1, L_2] = L_1 \times L_2$. Below we make use of the following instances of this result.

**Fact 1.1** ([5])

\[
(i) \quad S_5 \times S_5 = [S_5, S_5]
\]
\[
(ii) \quad S_5 \times K = [S_5, K]
\]
\[
(iii) \quad K \times K = [K, K]
\]
\[
(iv) \quad \text{Alt} \times \text{Alt} = [\text{Alt}, \text{Alt}]
\]
\[
(v) \quad \text{AltD} \times \text{AltD} = [\text{AltD}, \text{AltD}].
\]

In (i) and (v) confluence and commutativity become equivalent, and in (ii) commutativity implies confluence.

**Decidability of products.** In general, not much is known concerning the decidability of binary products. If none of the logics $L_1, L_2$ is $S_5$, then the only well-known products which are proved decidable in [5] are $D \times D$ and $K \times K$ (Theorems 10.18 and 9.13). In particular, decidability for $S_4 \times S_4$ is open. There are two extreme cases in which decision problems become much easier. For products with a functional modality, see below. For the case when one of the logics is $S_5$, Theorem 12.12 in [5] provides several decidable examples, for instance, all of $L \times S_5$ where $L \in \{K, D, T, K4, S_4, S_5\}$.

**Decision problems.** Connected with a semantically presented modal logic of a class of frames $K$ are several natural decision problems. We will study the following.

**Satisfaction problem:** given a formula $\varphi$, can $\varphi$ be satisfied in a state in a model over a frame in $K$?

**Global satisfaction problem:** given a formula $\varphi$, can $\varphi$ be satisfied in every state in a model over a frame in $K$?

**Validity problem:** given a formula $\varphi$, is $\varphi$ valid, that is does $\varphi$ hold in every state in every model on every frame in $K$?

**Global consequence problem:** given formulas $\varphi$ and $\psi$, does $\varphi$ globally entail $\psi$, that is, is $\psi$ true in every model on a $K$-frame where $\varphi$ is true? This is also called the valid rule problem (i.e., is $\varphi/\psi$ a $K$-valid inference rule).
As usual, we denote validity of \( \varphi \) in a class \( K \) of frames by \( K \models \varphi \). We use \( \varphi \models^K \psi \) to denote that \( \psi \) is a global consequence of \( \varphi \) in the class \( K \). The satisfaction problem and the validity problem are each others complement: \( \varphi \) is satisfiable iff \( \neg \varphi \) is not valid. The global satisfaction problem is partly connected to the global consequence problem: \( \varphi \) is globally satisfiable iff \( \varphi \) doesn’t globally entail \( \bot \).

The global satisfaction and consequence problem are not very often studied for modal logics, mainly because the emphasis is on the local consequence relation (\( \varphi \) locally entails \( \psi \) if in any model, every state where \( \varphi \) holds, is a state where \( \psi \) holds). With the local consequence relation, the implication trivially gives us a deduction theorem, so the validity and the consequence problems collapse into one. With the global consequence relation this need not be the case. In section 3 we will see examples of logics with a decidable satisfaction (whence validity) problem, but an undecidable global satisfaction (whence global consequence) problem.

The distinction also naturally arises when considering the standard translation of modal logic to first order logic. The satisfaction problem corresponds to satisfying the first order formula with one free variable, while the global satisfaction problem corresponds to satisfying the sentence obtained from the translation by universally quantifying over the free variable.

For practical applications of a logic it is often the complexity of the consequence relation that one is interested in. It might depend on the specific application whether this should be the global or the local one.

**Complexity.** Complexity analysis for modal logics is a well-developed field and we won’t attempt to review it here. Rather we will point to the relevant literature and very briefly recall some basic facts. For general computational complexity theory we refer to [16]. Classical papers on complexity of modal logics are [10, 3, 21, 9]. Especially the last one forms a good introduction for modal logicians. An excellent presentation of complexity in modal logic is given in the book [2]. This is also a good introduction to the more advanced [22].

We now briefly recall some basic facts, quoting freely from [9]. In this paper we are interested in the inherent difficulty of deciding for a given formula one of the above mentioned problems. Since all these problems are yes/no problems, we can view them as the difficulty of determining membership in a set. Thus the validity problem is viewed as the problem of determining whether a given formula \( \varphi \) is a member of the set of valid formulas with respect to a certain class \( K \). The difficulty of determining set membership is measured in terms of time and/or (memory)space required to do this, as a function of the size of the input, in our case the length of the formula \( \varphi \). Here we will mainly be concerned with time-complexity classes like \( P \), \( NP \), \( EXPTIME \) and \( NEXPTIME \): those sets such that determining whether a given element \( x \) is a member of the set can be done in deterministic polynomial time, nondeterministic polynomial time, deterministic exponential time, and nondeterministic exponential time, respectively (as a function of the size of \( x \)). These classes are ordered as \( P \subseteq NP \subseteq EXPTIME \subseteq NEXPTIME \), and it is known that \( P \neq EXPTIME \).

Given a complexity class \( C \), the class \( \text{co}-C \) consists of all the sets whose complement is a member of \( C \). It is easy to show that for deterministic classes \( C = \text{co}-C \), while it is believed that for non-deterministic (time)classes they are different. As mentioned above, satisfiability and validity are complementary problems, so determining that satisfiability is in \( C \) immediately implies that validity is in \( \text{co}-C \).

Roughly speaking, a set \( A \) is said to be **hard** with respect to a complexity class \( C \) if
every set in $C$ can be effectively reduced to $A$; i.e. for any set $B \in C$, an algorithm deciding membership in $B$ can be easily obtained from an algorithm deciding membership in $A$. For classes above $p$, easily means that there is a polynomial time computable function which takes any $B$ input $x$ and transforms it into an $A$-input $f(x)$ such that $x \in B$ iff $f(x) \in A$. A set is said to be complete with respect to $C$ if it is both in $C$ and $C$-hard.

Usually $p$ is considered the only tractable complexity class. For modal logicians this is infeasible since satisfiability of propositional logic is already NP-complete. The complexity of the satisfaction problem of most well-known modal logics lies between NP and EXPTIME, with many being polynomial space (PSPACE) complete. (E.g., S5 is NP-complete, all of $K,T,K4,S4$ are PSPACE-complete, and the expansion of modal logic $K$ with the universal modality is EXP-time-complete.) These space and time classes are related as follows: NP $\subseteq$ PSPACE $\subseteq$ EXPTIME, with all inclusions not known but generally believed to be strict.

2 Lower bounds

We will show that the satisfaction problem of several product logics is NEXPTIME-hard. A key element in the proof is the fact that we can force a model to contain a binary tree. Our proof is a reduction of the satisfaction problem to the tiling problem.

Tiling problems. [24] contains a modern introduction to tiling problems and gives examples of problems which are complete for the most common complexity classes arising in logic. A tile $T$ is a $1 \times 1$ square fixed in orientation with colored edges right($T$), left($T$), up($T$), and down($T$) taken from some denumerable set. A tiling problem takes the following form: given a finite set $\mathcal{T}$ of tile types, can we cover a certain part of $\mathbb{Z} \times \mathbb{Z}$, using only tiles of this type, in such a way that adjacent tiles have the same color on the common edge. We just list the tiling problems we will use. More can be found in the literature (e.g., [24]).

$\mathbb{N} \times \mathbb{N}$ tiling. Given a finite set $\mathcal{T}$ of tiles, can $\mathcal{T}$ tile $\mathbb{N} \times \mathbb{N}$?

$n \times n$ tiling. Given a set $\mathcal{T} = \{T_0, T_1, \ldots, T_t\}$ of tiles, can $\mathcal{T}$ tile $n \times n$ with the restriction that $T_0$ tiles the edges of $n \times n$?

Theorem 2.1 (i) $n \times n$-tiling (with $n$ given in unary) is NP-complete.
(ii) $n \times n$-tiling (with $n$ given in binary) is NEXPTIME-complete.
(iii) $\mathbb{N} \times \mathbb{N}$-tiling is undecidable, it is re-complete.

For proofs and history of these theorems we refer to [16].

Our first theorem shows that adding the confluence and commutativity axioms is in several cases enough to obtain a jump in complexity.

Theorem 2.2 Let $L$ be any normal extension of $[K,K]$ such that all models on finite S5 frames are models of $L$. Then the satisfaction problem of $L$ is NEXPTIME-hard.

From this theorem we immediately obtain the following

Corollary 2.3 Let $L_1$ and $L_2$ be any logics from the following list $K,D,T,K4,S4,S5$. The satisfaction problem of the product of $L_1$ and $L_2$ is NEXPTIME-hard.
PROOF. Let $L$ be as in the theorem. We will reduce the $n \times n$ tiling problem, with $n$ given in binary, to the satisfaction problem of $L$.

Given an instance $(k, \Sigma)$ of the tiling problem with $k$ given in binary, we will provide a formula which is satisfiable iff $(k, \Sigma)$ is a yes-instance. Without loss we may assume that $k = 2^n$ for some $n$. From now on let an instance $(2^n, \Sigma)$, where $\Sigma = \{T_0, \ldots, T_1\}$ be fixed.

We will define a formula $\varphi_{n, \Sigma}$ of length polynomial in $n$ and $|\Sigma|$ which describes this instance. In order to obtain the lower bound we show that

(A) $\varphi_{n, \Sigma}$ is computable in polynomial time from the instance $(2^n, \Sigma)$ (2$^n$ in binary),

(B) if $\Sigma$ tiles $2^n \times 2^n$, then $\varphi_{n, \Sigma}$ is satisfiable on a finite S5$^2$ model, and

(C) if $\varphi_{n, \Sigma}$ is $L$-consistent, then $\Sigma$ tiles $2^n \times 2^n$.

Given (A),(B) and (C) we have an effective reduction from the NEXPTIME-complete tiling problem to $L$-satisfiability, and the theorem follows.

We first describe how we represent a tile $T$ of the $2^n \times 2^n$-grid. A pair $(k, l) \in 2^n \times 2^n$ can be represented by a binary number of length $2n$; the first $n$ places for $k$ and the second $n$ for $l$. We now make a tile variable $t_i$ true at a leaf $x$ in the tree $(N, E)$ precisely if the tile $T_i$ tiles the pair $(k, l)$ whose representation is $x$, that is

$$(x, x) \in v(t_i) \iff T_i \text{ tiles } (k, l) \text{ and } (\forall i \leq n) x(i) = k(i) \text{ and } (\forall n < i \leq 2n) x(i) = l(i).$$

The $t_i$ variables obtain their valuation just as the $p_i^h$ and $p_i^v$.

We now describe $\varphi_{n, \Sigma}$ and show that it is satisfied at the pair $(a, a)$ in $(\Sigma, v)$, where $a$ is the root of the tree $(N, E)$.

We first describe a binary tree of depth $2n$, using the propositional variables $p_1, \ldots, p_{2n}$. This will provide us with $(2^n)^2$ leaves each encoding an element in the $2^n \times 2^n$ grid. Our
formula is similar to the one given in [9]: Proposition 6.5. (A variant of this formula was first considered in [10].)

Let $\diamond \varphi$ be an abbreviation for $\Phi(e \land \Diamond(d \land \varphi))$, and define $\Box \varphi = \neg \Box \neg \varphi$. We use $\diamond$ as an ordinary $\mathbf{K}$-modality. $\Box^n$ is an abbreviation defined as: $\Box^0 \varphi = \varphi$ and $\Box^{n+1} \varphi = \Box^n \Box \varphi$.

$$
d \land \Box^n \Box \bot \land_{k<2^n} \Box^k [ (\diamond p_{k+1} \land \Box \neg p_{k+1}) \land \bigwedge_{i \leq k} ((p_i \rightarrow \Box p_i) \land (\neg p_i \rightarrow \Box \neg p_i))]. \tag{3}
$$

Note that the leaves in such a tree make $d \land \Box \bot$ true. Clearly (3) is satisfied at $(a, a)$. The other formulas have a rather redundant formulation if we think about $\mathbf{S}5^2$ models. Since we want to prove the theorem for a wide class of logics, we have to use this particular formulation.

The next two formulas say that on any leaf, precisely one tile variable $t$ holds. Clearly they are satisfied at $(a, a)$.

$$
\Box^2 [d \land \Box \bot \leftrightarrow \bigvee_{1 \leq i \leq t} t_i] \tag{4}
$$

$$
\Box^2 [\bigwedge_{1 \leq i \leq t} (t_i \rightarrow \bigwedge_{j \neq i} \neg t_j)]. \tag{5}
$$

The following formula ensures that the tile $T_0$ is placed along the edges.

$$
\Box^2 [d \land \Box \bot \land (\neg p_1 \land \ldots \land \neg p_n) \lor (p_1 \land \ldots \land p_n) \lor (\neg p_{n+1} \land \ldots \land \neg p_{2n}) \lor (p_{n+1} \land \ldots \land p_{2n}) \rightarrow t_0] \tag{6}
$$

The next set of formulas capture the behaviour of the variables indexed by $h$ and $v$. First we write formulas ensuring inheriting of information: Let $x_i$ stand for any of $\{p_i, t_i, \neg p_i, \neg t_i\}$

$$
\Box^2 [\varphi^2 (x_i) \rightarrow x_i^h] \tag{7}
$$

$$
\Box^2 [\varphi^2 (x_i) \rightarrow x_i^v] \tag{8}
$$

Then we propagate the new variables in the right direction. Obviously $(\mathfrak{F}, v) \models (7)-(10)$.

$$
\Box^2 [x_i^h \rightarrow \Box^2 x_i^h] \tag{9}
$$

$$
\Box^2 [x_i^v \rightarrow \Box^2 x_i^v] \tag{10}
$$

Now we can express that colors match:

$$
\Box^2 [x_i^h = x_v \land y_v = y_v \land t_i^v \land \bigvee_{1 \leq j \leq t} t_j^h \rightarrow \bigvee_{1 \leq j \leq t} t_j^h \land \text{top}(T_i) = \text{bottom}(T_j^h)] \tag{12}
$$

$$
\Box^2 [y_i^h = y_v \land x_v = x_v \land t_i^v \land \bigvee_{1 \leq j \leq t} t_j^h \rightarrow \bigvee_{1 \leq j \leq t} t_j^h \land \text{right}(T_i) = \text{left}(T_j^h)] \tag{12}
$$

Here we use the following abbreviations:

$x_h = x_v$ abbreviates $\bigwedge_{i \leq n} (p_i^h \leftrightarrow p_i^v)$

$x_h = x_v + 1$ abbreviates $\bigvee_{i \leq n} [\bigwedge_{i \leq n} (p_i^h \leftrightarrow p_i^v) \land p_i^h \land \neg p_i^v \land \bigwedge_{i \leq n} (\neg p_i^h \land p_i^v)]$.

and similar for the $y$ coordinates where we use the $p_j$’s between $p_{n+1}$ and $p_{2n}$.

We show that (11) holds everywhere in the model $(\mathfrak{F}, v)$. Let $(k, l) \models x_h = x_v \land y_h = y_v + 1 \land t_i^h \land \bigvee_{1 \leq j \leq t} t_j^v$. Then both $k$ and $l$ are leaves in $(N, E)$ by the valuation of the $t_i$. Then $k$ encodes a pair $(x, y)$ and $l$ a pair $(x, y+1)$ in the grid. Because $(k, l) \models t_i^v$,
(k, k) \Vdash t_i$, whence $T_i$ tiles $(x, y)$. But since colors match $(x, y+1)$ must be tiled by a $T_j$ such that $\text{top}(T_i) = \text{bottom}(T_j)$. Then $(l, l) \Vdash t_j$ and indeed $(k, l) \Vdash t_j$ as desired.

So we have provided the formula $\varphi_{n, \mathcal{T}}$ and shown (B). Clearly the length of this formula is polynomial in $|\mathcal{T}|$ and $n$, and can be effectively obtained given $n$ and $\mathcal{T}$. This proves (A).

Finally we show that the given formula is indeed powerful enough to describe a tiling. So let $\varphi_{n, \mathcal{T}}$ be $L$-consistent. Then $\varphi_{n, \mathcal{T}}$ is satisfied in the canonical model $\mathfrak{M}$ for $L$. The frame underlying $\mathfrak{M}$ is a structure $\mathfrak{F} = (W, H, V)$ with $H$ and $V$ binary relations on $W$. Since $L$ is an extension of $[K, K]$, the frame satisfies (1) and (2) because commutativity and confluence are Sahlqvist-formulas. Now suppose that $\mathfrak{M}, w \Vdash \varphi_{n, \mathcal{T}}$. Then formula (3) forces a binary tree of depth $2n$ starting at $w$, in which the leaves encode all possible valuations of the $p_i$-variables. By (1), every leaf in that tree can be reached from $w$ by making $2n$ horizontal, followed by $2n$ vertical steps. So (4) and (5) ensure that at each leaf precisely one tile variable holds. Choose one such a tree starting at $w$ and define a tiling of the grid, using the encoding given above. By (4) and (5) this is a well-defined tiling. By (6), the tile $T_0$ is placed along the edges. Now we check that colors match. Suppose that $T_i$ tiles $(x, y)$ and $T_j$ tiles $(x, y+1)$. Then by commutativity we have the following situation in our model:

$$wV^{2n}aH^{2n}l\land wH^{2n}bV^{2n}k$$

for some worlds $a$ and $b$, where $k$ is a leaf encoding $(x, y)$ and $\mathfrak{M}, k \Vdash t_i$, and $l$ a leaf encoding $(x, y+1)$ and $\mathfrak{M}, l \Vdash t_j$. By the inheritance formulas (7) and (8) we have $\mathfrak{M}, a \Vdash (x, y+1)^v \land t_j^v$ and $\mathfrak{M}, b \Vdash (x, y)^v \land t_i^v$, where $(x, y+1)^v$ abbreviates the conjunction of $p_i^h$ and $\neg p_i^h$ such that $p_i$ and $\neg p_i$ are true at $l$, etcetera. Now by confluence we have a world $c$ such that $aH^{2n}c$ and $bV^{2n}c$, so $\mathfrak{M}, c \Vdash (x, y+1)^h \land t_j^h \land (x, y)^v \land t_i^v$ by the propagation formulas (9) and (10). But then by (11) the colors must indeed match. The same argument with (12) shows that colors match horizontally. This proves (C), whence the theorem.

QED

In the next section we show for $S5 \times S5$ and $S5 \times K$ a matching upper bound. If one of the logics contains the axiom $\Diamond p \rightarrow \Box p$ (its accessibility relation is a partial function), then Theorem 2.2 does not apply, and indeed we can find much better upper bounds, cf. below. But before we look at the upper bounds we briefly consider the complexity of the global satisfaction problem.

2.1 Global satisfaction problem

We will now consider the question of global satisfaction for the logics $\text{Alt}^2$, $K^2$ and $D^2$. As mentioned before, the satisfaction problem of them is decidable, for $\text{Alt}^2$ it is even in $\text{NP}$ (cf. Theorem 3.7 below). With global satisfaction the problem is quite a bit harder. For modal logic $K$, the satisfaction problem is $\text{pspace}$-complete by [10], and the global satisfaction problem is $\text{exptime}$-complete ([22] cf. [12] how the lower bound can be derived from the result about the expansion of $K$ with the universal modality).

**Theorem 2.4** The global satisfaction problems of $\text{Alt}^2$, $K^2$ and $D^2$ are undecidable.

**Proof.** We start with the easy $\text{Alt}^2$ by encoding $\mathbb{N} \times \mathbb{N}$-tiling. Let $\mathcal{T} = \{T_1, \ldots, T_k\}$ be a set of tiles. Let $t_1, \ldots, t_k$ be propositional variables. Define $\varphi_{\mathcal{T}}$ as the conjunction of the following formulas:
$$AX_0 \quad \phi \top \land \boxdot \top$$

$$AX_1 \quad \bigvee_{1 \leq i \leq k} t_i$$

$$AX_2 \quad t_i \rightarrow \bigwedge_{i \neq j} \neg t_j$$

$$AX_3 \quad t_i \rightarrow \Box \bigvee \{ t_j \mid \text{right}(T_i) = \text{left}(T_j) \}$$

$$AX_4 \quad t_i \rightarrow \Box \bigvee \{ t_j \mid \text{top}(T_i) = \text{bottom}(T_j) \}.$$ 

It is almost immediate that $\exists$ tiles $\mathbb{N} \times \mathbb{N}$ if and only if there exists a $\text{Alt}^2$ model where $\varphi_\exists$ holds everywhere (the accessibility relations are the two successor functions in the grid).

For $K^2$ we show that $\varphi_\exists$ is globally $\text{Alt}^2$-satisfiable iff $\varphi_\exists$ is globally $K^2$-satisfiable, which by the above reduction to the tiling problem is sufficient. From left to right is trivial since every $\text{Alt}^2$-model is also a $K^2$-model. For the other direction, suppose $\mathfrak{M} = (W, H, V, \nu) \models \varphi_\exists$ and $\mathfrak{M}$ is a $K^2$-model. Because $\phi \top \land \boxdot \top$ holds, every world has both $H$ and $V$ successors. Choose a substructure of $(W, H, V)$, where $H$ and $V$ are commuting total functions (this can easily be shown to exist). Let $\mathfrak{M}^*$ be the submodel of $\mathfrak{M}$ whose frame is this substructure.

By the special form of $\varphi_\exists$, its global truth is preserved under taking substructures, whence also $\mathfrak{M}^* \models \varphi_\exists$. Clearly $\mathfrak{M}^*$ is an $\text{Alt}^2$-model.

Obviously the same proof works for $D^2$. QED.

Since a formula $\varphi$ is globally satisfiable iff $\varphi \not\models^* \bot$, we obtain immediately the following

**Corollary 2.5** The global consequence problems of $\text{Alt}^2$, $K^2$ and $D^2$ are undecidable.

Many logics—the most prominent is first order logic—have a way of coding the global consequence problem as a validity problem. The route by which this coding works is the deduction theorem.

Let $L$ be a modal logic. We say that $L$ has a deduction theorem if there exists an $L$-formula $f(p, q)$ in two variables $p$ and $q$ such that $\varphi \models^* \psi$ in $L$ iff $f(\varphi, \psi)$ is $L$-valid.

Clearly when a modal logic has a deduction theorem, the satisfaction problem and the global satisfaction problem, and hence the validity and the valid consequence problem collapse: they will have the same complexity.

Any modal logic extending $K_4$ has a deduction theorem. When one defines $f(p, q)$ as $(p \land \Box p) \rightarrow q$, a simple generated subframe argument suffices to show this. The next theorem says that this behaviour is preserved when taking products.

**Theorem 2.6** Let $L_1, L_2$ both stronger than $K_4$. Then the product of $L_1$ and $L_2$ has a deduction theorem.

**Proof.** Define $\Diamond \varphi$ as $\phi \lor \varphi \lor \lozenge \varphi \lor \Phi \varphi$. Then $\Diamond$ is a $S_4$-modality because $\Phi$ and $\lozenge$ are both $K_4$, and they commute in the product. Moreover by definition of $\Diamond$, we have $\Diamond \varphi \rightarrow \Diamond \Diamond \varphi$ and $\Diamond \varphi \rightarrow \Diamond \varphi$. Now the same definition of $f(p, q)$ as with $K_4$ can easily be shown to work, again with a generated subframe argument. QED.

### 3 Upper bounds

In this section, we look at two special cases of binary products. First we consider products with an $S_5$ modality, and then products with a functional modality. In the former case, we find upper bounds matching the lower bounds of the previous section. In the latter case, we see that taking products does not increase complexity.
3.1 Products with an $S5$ modality

As mentioned in the introduction [5] contains several decidability results for products with an $S5$ modality. They show for instance that for any of the logics $S5 \times L$ with $L \in \{K, T, D, K4, S4, S5\}$ every satisfiable formula $\varphi$ can be satisfied in a $[S5, L]$ model of size double exponential in the length of $\varphi$. This gives a rather bad non-deterministic double-exponential time upper bound because for all these logics $[S5, L] = S5 \times L$. This result is proved using an adaptation of the filtration technique. Better bounds can be obtained using mosaic style construction methods, as proposed in [18] for the product of $S5$ and the temporal logic of linear orders. This logic does not even have the finite model property, but it is shown to have a decision algorithm which runs in deterministic double-exponential time. We will use this technique for the product $S5 \times K$.

For $S5^2$, the upper complexity bound can be derived from the NEXPTIME upper bound of first order logic with two variables [6] using the translation provided in the introduction. A direct mosaic-style proof is provided in [13]. So together with Theorem 2.2 we obtain

**Corollary 3.1** The satisfaction problem of $S5^2$ is NEXPTIME-complete.

We will now show that the satisfaction problem of $S5 \times K$ is in NEXPTIME as well, using ideas from [18]. In what follows we consider the product $S5 \times K$, whence $\Box$ is an $S5$, and $\Phi$ a $K$-modality. Since $[S5, K] = S5 \times K$ by Fact 1.1, we can satisfy formulas both in $S5 \times K$ product frames and in $[S5, K]$-Kripke models over frames of the form $(W, H, V)$ where $H$ is an equivalence relation and $H$ and $V$ satisfy the commutative law (1). The main idea is that we can view a frame as being constructed from rows, now called segments.

**Definition 3.2** Fix a formula $\xi$. $Cl_i(\xi)$ denotes the set of all subformulas of $\xi$ plus single negations of modal depth less than $i$. $d(\varphi)$ denotes the modal depth of $\varphi$. For $x$ a world in a model $\mathfrak{M}$, $[x]_{Cl_i(\xi)}$ denotes the set $\{\psi \in Cl_i(\xi) | \mathfrak{M}, x \models \psi\}$ of all formulas in $Cl_i(\xi)$ which are true at $x$ in $\mathfrak{M}$.

A segment is a structure $(X, i)$, with $i$ a natural number and $X \subseteq P(Cl_i(\xi))$.

A segment $(X, i)$ is called coherent if for all $x \in X$, it satisfies

- $FC^< \varphi \in x \leftrightarrow \neg \varphi \notin x$ provided $\neg \varphi \in Cl_i(\xi)$
- $FC^\lor \varphi \land \psi \in x \leftrightarrow \varphi \in x$ and $\psi \in x$ provided $\varphi \land \psi \in Cl_i(\xi)$
- $FC^\top \forall \varphi \in x \leftrightarrow$ there exists a $y \in X$ and $\varphi \in y$ provided $\Box \varphi \in Cl_i(\xi)$.

A set $S$ of segments is called saturated if whenever $(X, i) \in S$ and $\Phi \varphi \in x$ for some $x \in X$, there exists a $(Y, i - 1) \in S$ with $a \in Y$ such that $\varphi \in y$ and a relation $B \subseteq X \times Y$ satisfying

- $FS1$ \hspace{1em} xBy
- $FS2$ \hspace{1em} the domain of $B$ is $X$ and its range is $Y$
- $FS3$ \hspace{1em} if $aBb$, then $\neg \Phi \psi \in a$ implies $\neg \psi \in a$.

A set $S$ of coherent segments is a saturated set of segments for $\xi$ (short: a $\xi$-SSS) if $S$ is saturated and there is an $(X, i) \in S$ with an $x \in X$ and $\xi \in x$.

**Lemma 3.3** If there exists a $\xi$-SSS, then $\xi$ is satisfiable. In particular, then $\xi$ is satisfiable in a $[S5, K]$ Kripke model whose set of worlds consists of all elements of all segments from the SSS.
Proof. Let \( M \) be the \( \xi \)-SSS. For \( S = (X, i) \in M \), define \( S' = \{(X, i)\} \times X \). Define a model \( \mathfrak{M} = (W, H, V, v) \) as follows:

\[
W = \bigcup \{S' \mid S \in M\}
\]

\[
H = \{(((X, i), \Delta), ((Y, j), \Gamma)) \mid (X, i) = (Y, j)\}
\]

\[
\nu(p) = \{((X, i), \Delta) \mid p \in \Delta\}, \text{ and}
\]

\[
((X, i), \Delta)V((Y, j), \Gamma) \text{ if } i = j + 1 \text{ and there exists a relation } B \subseteq X \times Y \text{ satisfying } FS_2, FS_3 \text{ and } \Delta B \Gamma.
\]

By definition \( H \) is an equivalence relation, and \( H \) and \( V \) commute by the definition of \( V \) and the condition \( FS_2 \), so \((W, H, V)\) is an \([S5, K]\) Kripke frame.

That \( \mathfrak{M} \) satisfies \( \xi \) follows from

for all \( ((X, i), \Delta) \in W \), for all \( \psi \in Cl_i(\xi) \), \( \mathfrak{M}, ((X, i), \Delta) \models \psi \iff \psi \in \Delta \),

which we prove by an induction on the complexity of \( \psi \). The propositional case goes through by definition, the boolean cases by \( FC^- \) and \( FC^\wedge \). The \( \Rightarrow \)-case is immediate by \( FC^\Phi \) and our definition of \( H \). The \( \Rightarrow \)-direction for the \( \Phi \)-case follows from condition \( FS_3 \), and the \( \Leftarrow \)-direction by saturatedness of the SS, together with the definition of \( V \).

QED

**Lemma 3.4** If \( \xi \) is \( S5 \times K \)-satisfiable, then there exists a \( \xi \)-SSS of size exponential in \(|\xi|\).

**Proof.** Let \( \mathfrak{M}, (a, b) \models \xi \), with \( \mathfrak{M} = (W_0 \times W_1, H, V, v) \). We assume \( \mathfrak{M} \) is generated from \((a, b)\). We will now take segments out of \( \mathfrak{M} \) in stages, very much like the K-world algorithm of Ladner and Spaan [10, 23].

**Stage 0.** Take the segment horizontally generated from \((a, b)\). That is

\[
SS_0 = \{\{[c, b]\}_{\overline{Cl_d(\xi)}} \mid aHc \text{ and } c \in W_0\}, d(\xi))\}.
\]

**Stage \( n + 1 \).** \( SS_{n+1} \) is defined as the least set of segments of the form \((X, d(\xi) - (n + 1))\) such that for every segment \((X, i) \in SS_n\), for every \([x, y]\)_{\overline{Cl_d(\xi)}} \subseteq X, for every \( \Phi \psi \in ([x, y])_{\overline{Cl_d(\xi)}}\), there is a segment

\[
\{\{z, w\}_{\overline{Cl_{d(\xi)-(n+1)}}(\xi)} \mid xHz \text{ and } z \in W_0\}, d(\xi) - (n + 1)\} \in SS_{n+1}
\]

containing an element \([x, w]\)_{\overline{Cl_{d(\xi)-(n+1)}}(\xi)} such that \( \mathfrak{M}, (x, w) \models \psi \) and \( yV w \).

**Stage \( d(\xi) + 1 \).** Define \( SS \) as the union of all sets of segments from the previous stages.

We claim that the set \( SS \) is a saturated set of segments for \( \xi \). All segments in \( SS \) are coherent because they come from a model. In stage 0, we included a segment containing \( \xi \), and in the inductive steps we included enough segments to guarantee saturatedness. Again, since the segments come from a model, conditions \( FS_2 \) and \( FS_3 \) are satisfied.

The size of a segment is exponential in \(|\xi|\). The number of segments in the set \( SS \) is also exponential in \( \xi \), since in each stage, for every segment \( S \in SS_n \) we only included at most \(|\xi| \cdot 2^{\xi} \) many new segments, and because the construction stops after at most \(|\xi| \) stages. So indeed, the obtained \( \xi \)-SSS has size exponential in \(|\xi|\).

QED

**Theorem 3.5** The satisfaction problem of \( S5 \times K \) is NEXPTIME-complete.

\[ \text{thm:1up} \]
Proof. The lower bound is provided by Theorem 2.2. We can now show the upper bound in two ways. Combining the previous two lemmas shows that every satisfiable formula can be satisfied in a \([S5, K]\)-model of exponential size in that formula. So we can guess a model of that size and check (this takes just polynomial time in the size of the model and the formula) whether it is an \([S5, K]\) model in which the formula is satisfied. Clearly this procedure is in \textit{nexptime}. Alternatively we can guess an exponential set of segments and check whether it is a \(\xi\)-SSS. In both cases, we use that \([S5, K] = S5 \times K\), provided by Fact 1.1.

3.2 Products with functions

The upper complexity bound of a logic does transfer when we multiply it with the modal logic of a \textit{functional} modality. This answers question 25 in [5]. So we obtain a gain in expressive power without an increase in complexity. This result might have applications in the theory of two-dimensional temporal logics, where one often sees very bad complexity results. If one restricts one dimension to \textit{tomorrow logic}, then the two-dimensional system inherits the upper bound from the other temporal dimension. The proof crucially depends on the fact that we can transform any formula into a formula without interaction of the modalities.

Let \(K\) be a class of modal frames, and let \(F\) be the class of frames \((W, f)\) with \(f\) a total function from \(W\) to \(W\). We denote the \(K\)-modality by \(\Diamond\) and the functional modality by \(\langle f \rangle\).

A formula in the product language is an \(f\)-formula if it is of the form \(\langle f \rangle^n \phi\), where \(\phi\) is a propositional variable or its negation. We say that a formula in this language is in \textit{normal form} if it is constructed from variables and \(f\)-formulas using \(\land\), \(\neg\) and \(3\).

Proposition 3.6 There exists a linear algorithm transforming every \(\{\Diamond, \langle f \rangle\}\)-formula \(\phi\) into an equivalent formula \(\phi^*\) in normal form.

Proof. Transform \(\phi\) using the following rewrite rules:

\[
\langle f \rangle \Diamond \psi \quad \Rightarrow \quad \Diamond \langle f \rangle \psi \\
\neg \langle f \rangle \psi \quad \Rightarrow \quad \langle f \rangle \neg \psi \\
\langle f \rangle (\phi \land \psi) \quad \Rightarrow \quad \langle f \rangle \phi \land \langle f \rangle \psi.
\]

Since these rules are \(K \times F\) valid in both directions, this transformation preserves \(K \times F\)-validity.

QED

Theorem 3.7 (i) Let \(F\) be the class of total function frames, and let \(K\) be the class of frames of a logic whose satisfaction problem is complete for complexity class \(C\). Then the satisfaction problem of the bi-modal logic of the product \(K \times F\) is also complete for \(C\).

(ii) The same as (i), but now for the class \(PF\) of partial function frames.

Proof. (i). The lower bound is immediate. For the upper bound, we first transfer an input formula into normal form. By the previous proposition this can be done in linear time. Now we transform this formula \(\phi\) into a pure \(K\)-formula \(\phi^*\) by replacing every \(f\)-formula \((f)^n \psi\) occurring in \(\phi\) by a variable \(p(f)^n \psi\). We claim that \(\phi\) is \(K \times F\)-satisfiable iff \(\phi^*\) is \(K\)-satisfiable in a model \((W, R, \nu)\) such that for every world \(w \in W\), the conjunction of formulas from the following set is \(F\)-satisfiable:

\[
\begin{align*}
\{p \mid w \in \nu(p) \text{ and } p \neq p(f)^n \psi\} & \quad \cup \\
\{-p \mid w \not\in \nu(p) \text{ and } p \neq p(f)^n \psi\} & \quad \cup \\
\{(f)^n \psi \mid w \in \nu(p(f)^n \psi)\} & \quad \cup \\
\{\neg (f)^n \psi \mid w \not\in \nu(p(f)^n \psi)\}
\end{align*}
\]

thm:funk

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For the right to left direction, take the product of $(W, R)$ and the natural numbers with successor. Let $v'$ be the smallest valuation satisfying

$$(w, 0) \in v'(p) \iff w \in v(p), \text{ for all } p \not\in p_{(f)\psi}$$

$$(w, n) \in v'(q) \iff w \in v(p_{(f)\psi}) \text{ or } w \not\in v(p_{(f)\psi})$$

By our assumption this valuation is well-defined, and clearly $\varphi$ is satisfied in this $K \times F$-model.

For the other direction, suppose $\varphi$ is satisfied at $(x, y)$ in the model $\mathfrak{M} = (W_1 \times W_2, R, f, v)$. Let $\mathfrak{M}^* = (W_1, R, v^*)$ where $v^*(p) = \{ w \in W_1 \mid (w, y) \in v(p) \}$ for all $p \not\in p_{(f)\psi}$ and $v^*(p_{(f)\psi}) = \{ w \in W_1 \mid \mathfrak{M}, (w, y) \models (f)\psi \}$. Clearly $\mathfrak{M}^*, x \models \varphi^*$, in the desired way. This proves (i).

For (ii) the lower bound is again immediate. For the upper bound we interpret the logic with the partial function modality. Let $(\cdot)^t$ from the $\{(pf), \diamond \}$ language to the $\{(f), \diamond \}$ language be defined as

\[
\begin{align*}
(p)^t & = p \\
(\neg \varphi)^t & = \neg \varphi^t \\
(\varphi \land \psi)^t & = \varphi^t \land \psi^t \\
(\varphi \lor \psi)^t & = \varphi^t \lor \psi^t \\
((pf)\varphi)^t & = (f)(w \land \varphi^t),
\end{align*}
\]

where $w$ is a new variable. Let $d(\varphi)$ denote the modal depth of $\varphi$, and let $\Box^{(n)}\varphi$ abbreviate $\varphi \land \Box \varphi \land \ldots \land \Box^n \varphi$. Let $D(\varphi)$ then stand for

\[
\bigwedge_{i \leq d(\varphi)} ((f)^i w \rightarrow \Box^{d(\varphi)}(f)^i w) \land ((f)^i \neg w \rightarrow \Box^{d(\varphi)}(f)^i \neg w).
\]

(13)

We claim that

$$\varphi$$ is $K \times PF$-satisfiable iff $D(\varphi) \land w \land \varphi^t$ is $K \times F$-satisfiable.

(14)

Note that $D(\varphi)$ is polynomial in $\varphi$, and $(\cdot)^t$ is a ptime reduction, so (14) indeed proves the theorem. For the left to right direction of (14), suppose $\varphi$ is satisfied in a $K \times PF$-model $\mathfrak{M} = (W_1 \times W_2, R, p f, v)$. (We assume $\mathfrak{M}$ is generated.) We will create a model $\mathfrak{M}^*$, giving a valuation for the new variable $w$. If $p f$ is total, we take the same frame and set $w$ true everywhere. If $p f$ is partial, then let $x$ be the world without successor in $W_2$. Define $\mathfrak{M}^* = (W_1 \times W_2^*, R, f, v^*)$, where $W_2^* = W_2 \cup \{ \text{new} \}$, $f = p f \cup \{(x, \text{new}), (\text{new}, \text{new})\}$, $v^* = v$ for all variables except $w$, and $v^*(w) = W_1 \times W_2$. Clearly $\mathfrak{M}^*$ satisfies the required formula.

For the other direction, let $D(\varphi) \land w \land \varphi^t$ be satisfied in a $K \times F$-model $\mathfrak{M} = (W_1 \times W_2, R, f, v)$ at the pair $(x, y)$ (from which we assume the model is generated). By $D(\varphi)$ we know that for all $x'$, for all $k \leq d(\varphi)$, such that $xR^kx'$, it holds that

$$x, f^k y \models w \iff (x', f^k y) \models w.$$

(15)

Let $(x, f^k y)$ be the first $\neg w$ world reachable from $(x, y)$, and let $W^*_2$ be $\{y, f y, \ldots, f^{k-1} y\}$. (If there are no $f$-reachable $\neg w$ worlds, then $W^*_2 = W_2$.) There are two cases. If $k > d(\varphi)$, then $\varphi$ is satisfied at $(x, y)$ in $\mathfrak{M}$ by (15) and because the value of $\varphi$ depends only on successors less than its modal depth away. If $k \leq d(\varphi)$, let $\mathfrak{M}^*$ be the submodel of $\mathfrak{M}$ with domain $W_1 \times W^*_2$, with $f(f^{k-1} y)$ undefined By (15), all pairs reachable in less than $d(\varphi)$ steps from $(x, y)$ validate $w$, so $\mathfrak{M}^*, (x, y) \models \varphi$.

QED
4 Related work and further directions

Section 15 in [5] contains a wealth of open questions (28 in total) about products of modal logics, several related to the work presented here. Below we list some of these and some additional ones.

Craig interpolation  For the computational aspects of this property, in particular with respect to modularisation of programs, we refer to [11, 17]. In [14] it is shown that interpolation fails in every logic of Theorem 2.2, but that $\text{AltD}^2$ has interpolation. [19]: Theorem 2 implies that the Beth definability property fails for $S5^2$.

Question 4.1 (i) For which logics does interpolation or the Beth-property transfer when taking products?
(ii) In particular, does every logic $[\text{AltD}, L]$ where $L$ is the logic of a Sahlqvist axiomatisable universal Horn class of frames have interpolation?
(iii) How far can Sain’s result be strengthened?
(iv) Do the answers differ when we consider tense-similarity types?

[18] contains the logic $S5 \times \text{Lin}$, where $\text{Lin}$ is the temporal logic of the class of linear orders.

Question 4.2 (i) What is the exact complexity of $S5 \times \text{Lin}$? By Theorem 2.2 it is $\text{NEXPTIME}$-hard and by Reynolds’ result it is in $2\text{-EXPTIME}$. We conjecture it is $\text{EXPSPACE}$-complete.
(ii) (Reynolds) Is $Df \times \text{Lin}$, where $Df$ is the class of frames for the difference operator, decidable?
(iii) What is the complexity of $Df^2$? By results of [12] and [7] the problem is decidable, but no upper bound can be obtained from either of these papers.

Question 4.3 (i) What is the complexity of $S4 \times S5$?
(ii) Is $K4^2$ decidable? If so, what is the complexity?
(iii) What is the complexity of $K^2$. Is it $\text{NEXPTIME}$-complete?

Question 4.4 Are there other non-trivial logics with the property that multiplying them with logics of complexity $C$ yields a product logic which is $C$-complete (as with the logic of a functional modality)?

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References


